

SMOOTHED DIFFERENTIATION FILTERS

Reference:

P. Meer, I. Weiss: Smoothed differentiation filters for images.
Journal of Visual Communication and Image Representation. **3**, 58–72, 1992.

The filters provide the optimal (in least squares sense), closed form solution to the problem:

Discrete data defined on the regular one-dimensional grid $i = -n, \dots, 0, \dots, n$, is assumed to represent samples of a degree p polynomial additively corrupted by zero-mean measurement noise. Estimate in $i = 0$ the value of the r -th ($r \leq p$) derivative of the underlying polynomial.

The filters are obtained employing orthogonal polynomial bases defined over a discrete interval. Chebyshev polynomials yield the filters for unweighted data, Krawtchouk polynomials yield the filters for data weighted with binomial weights. Note that the filters are valid only for the regular sampling grid which is a necessary condition for the orthogonality of the polynomials to hold.

The sequence $h(i; r, p)$, $i = -n, \dots, 0, \dots, n$, is the filter for estimating the r -th derivative when a degree- p polynomial is assumed for the underlying structure. The filter is applied as

$$output = \sum_{i=-n}^n h(i; r, p) * input(i)$$

Some important properties:

- The same filter is obtained for two consecutive degrees of the underlying polynomial. For any given r and p such that $\text{mod}(r + p, 2) = 0$, $h(i; r, p) \equiv h(i; r, p + 1)$.
- $h(-i; r, p) = (-1)^r h(i; r, p)$
- When the input consist of the uncorrupted samples of a polynomial (up to degree p), the output is the theoretical value, i.e., it is not distorted.
- The smoothing filters, i.e., $h(i; 0, p)$,
 - preserve the first p moments of the true (uncorrupted) input
 - achieve maximal (in least squares sense) noise rejection.

Combining two filters in a dyadic product provides 2D window operators. For example, weighting the data (using filters derived from the Krawtchouk polynomials) and smoothing along one coordinate with constant/linear underlying structure, while computing the first derivative along the other coordinate with linear/quadratic structure, yields an edge detector very similar to the widely used implementation of the Canny operator.

Unweighted Data

The filters $h_C(i; r, p)$ are built using the Chebyshev polynomials, p being the smaller of the two polynomial degrees yielding identical sequences.

Smoothing. $r = 0$. $p = 0$ or 1 .

$$h_C(i; 0, 0) = \frac{1}{2n + 1}$$

Smoothing. $r = 0$. $p = 2$ or 3 .

$$h_C(i; 0, 2) = -\frac{3[5i^2 - (3n^2 + 3n - 1)]}{(2n - 1)(2n + 1)(2n + 3)}$$

Smoothing. $r = 0$. $p = 4$ or 5 .

$$h_C(i; 0, 4) = \frac{15[63i^4 - 35(2n^2 + 2n - 3)i^2 + (15n^4 + 30n^3 - 35n^2 - 50n + 12)]}{4(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5)}$$

First derivative. $r = 1$. $p = 1$ or 2 .

$$h_C(i; 1, 1) = \frac{3i}{n(n + 1)(2n + 1)}$$

First derivative. $r = 1$. $p = 3$ or 4 .

$$h_C(i; 1, 3) = -\frac{5[7(3n^2 + 3n - 1)i^3 - 5(3n^4 + 6n^3 - 3n + 1)i]}{(n - 1)n(n + 1)(n + 2)(2n - 1)(2n + 1)(2n + 3)}$$

First derivative. $r = 1$. $p = 5$ or 6 .

$$\begin{aligned} h_C(i; 1, 5) = & \left(\frac{21}{4}\right) \frac{33(15n^4 + 30n^3 - 35n^2 - 50n + 12)i^5 -}{(n - 2)(n - 1)n(n + 1)} \\ & - \frac{105(6n^6 + 18n^5 - 15n^4 - 60n^3 + 17n^2 + 50n - 12)i^3 +}{(n + 2)(n + 3)(2n - 3)(2n - 1)} \\ & + \frac{7(25n^8 + 100n^7 - 50n^6 - 500n^5 - 95n^4 + 760n^3 + 180n^2 - 300n + 72)i}{(2n + 1)(2n + 3)(2n + 5)} \end{aligned}$$

Second derivative. $r = 2$. $p = 2$ or 3 .

$$h_C(i; 2, 2) = \frac{30[3i^2 - n(n+1)]}{n(n+1)(2n-1)(2n+1)(2n+3)}$$

Second derivative. $r = 2$. $p = 4$ or 5 .

$$h_C(i; 2, 4) = - \left(\frac{105}{2} \right) \frac{15(6n^2 + 6n - 5)i^4 - 21(4n^4 + 8n^3 - 4n^2 - 8n + 5)i^2 + 5(n-1)n(n+1)(n+2)(2n^2 + 2n - 3)}{(n-1)n(n+1)(n+2)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}$$

Third derivative. $r = 3$. $p = 3$ or 4 .

$$h_C(i; 3, 3) = \frac{210[5i^3 - (3n^2 + 3n - 1)i]}{(n-1)n(n+1)(n+2)(2n-1)(2n+1)(2n+3)}$$

Third derivative. $r = 3$. $p = 5$ or 6 .

$$h_C(i; 3, 5) = - \left(\frac{945}{2} \right) \frac{77(2n^2 + 2n - 3)i^5 - 15(12n^4 + 24n^3 - 28n^2 - 40n + 39)i^3 + 7(6n^6 + 18n^5 - 15n^4 - 60n^3 + 17n^2 + 50n - 12)i}{(n-2)(n-1)n(n+1)(n+2)(n+3)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}$$

Fourth derivative. $r = 4$. $p = 4$ or 5 .

$$h_C(i; 4, 4) = \frac{1890[35i^4 - 5(6n^2 + 6n - 5)i^2 + 3(n-1)n(n+1)(n+2)]}{(n-1)n(n+1)(n+2)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}$$

Fifth derivative. $r = 5$. $p = 5$ or 6 .

$$h_C(i; 5, 5) = \frac{20790[63i^5 - 35(2n^2 + 2n - 3)i^3 + (15n^4 + 30n^3 - 35n^2 - 50n + 12)i]}{(n-2)(n-1)n(n+1)(n+2)(n+3)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)}$$

Weighted Data

The filters $h_K(i; r, p)$ are built using the Krawtchouk polynomials, p being the smaller of the two polynomial degrees yielding identical sequences. The binomial weights

$$w(i) = \frac{1}{2^{2n}} \binom{2n}{n+i} = \frac{1}{2^{2n}} \frac{(2n)!}{(n-i)!(n+i)!}$$

are common to all the filters.

Smoothing. $r = 0$. $p = 0$ or 1 .

$$h_K(i; 0, 0) = w(i)$$

Smoothing. $r = 0$. $p = 2$ or 3 .

$$h_K(i; 0, 2) = -\frac{2i^2 - (3n - 1)}{2n - 1} w(i)$$

Smoothing. $r = 0$. $p = 4$ or 5 .

$$h_K(i; 0, 4) = \frac{4i^4 - 20(n-1)i^2 + (15n^2 - 25n + 6)}{2(2n-3)(2n-1)} w(i)$$

First derivative. $r = 1$. $p = 1$ or 2 .

$$h_K(i; 1, 1) = \frac{2i}{n} w(i)$$

First derivative. $r = 1$. $p = 3$ or 4 .

$$h_K(i; 1, 3) = -\frac{2[2(3n-1)i^3 - (15n^2 - 15n + 4)i]}{3(n-1)n(2n-1)} w(i)$$

First derivative. $r = 1$. $p = 5$ or 6 .

$$h_K(i; 1, 5) = \left(\frac{1}{15} \right) \frac{4(15n^2 - 25n + 6)i^5 - 20(21n^3 - 63n^2 + 56n - 12)i^3 + (525n^4 - 2100n^3 + 2835n^2 - 1480n + 276)i}{(n-2)(n-1)n(2n-3)(2n-1)} w(i)$$

Second derivative. $r = 2$. $p = 2$ or 3 .

$$h_K(i; 2, 2) = \frac{4[2i^2 - n]}{n(2n - 1)}w(i)$$

Second derivative. $r = 2$. $p = 4$ or 5 .

$$h_K(i; 2, 4) = -\left(\frac{4}{3}\right) \frac{4(3n - 2)i^4 - 2(24n^2 - 39n + 17)i^2 + 15n(n - 1)^2}{(n - 1)n(2n - 3)(2n - 1)}w(i)$$

Third derivative. $r = 3$. $p = 3$ or 4 .

$$h_K(i; 3, 3) = \frac{8[2i^3 - (3n - 1)i]}{(n - 1)n(2n - 1)}w(i)$$

Third derivative. $r = 3$. $p = 5$ or 6 .

$$h_K(i; 3, 5) = -\frac{8[4(n - 1)i^5 - 2(12n^2 - 27n + 16)i^3 + (21n^3 - 63n^2 + 56n - 12)i]}{(n - 2)(n - 1)n(2n - 3)(2n - 1)}w(i)$$

Fourth derivative. $r = 4$. $p = 4$ or 5 .

$$h_K(i; 4, 4) = \frac{16[4i^4 - 4(3n - 2)i^2 + 3(n - 1)n]}{(n - 1)n(2n - 3)(2n - 1)}w(i)$$

Fifth derivative. $r = 5$. $p = 5$ or 6 .

$$h_K(i; 5, 5) = \frac{32[4i^5 - 20(n - 1)i^3 + (15n^2 - 25n + 6)i]}{(n - 2)(n - 1)n(2n - 3)(2n - 1)}w(i)$$